Exercise 24

Prove the formulas given in Table 1 for the derivatives of the functions (a) cosh, (b) tanh, (c) csch, (d) sech, and (e) coth.

Solution

Write each of the hyperbolic functions in terms of exponential functions in order to evaluate the derivatives.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})}{2} = \frac{e^x - e^{-x}\cdot(-1)}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{\frac{d}{dx}(e^x) + \frac{d}{dx}(e^{-x})}{2} = \frac{e^x + e^{-x}\cdot(-1)}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\begin{aligned}\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) &= \frac{\left[\frac{d}{dx}(\sinh x)\right]\cosh x - \left[\frac{d}{dx}(\cosh x)\right]\sinh x}{\cosh^2 x} \\ &= \frac{(\cosh x)\cosh x - (\sinh x)\sinh x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) &= \frac{\left[\frac{d}{dx}(1)\right]\sinh x - \left[\frac{d}{dx}(\sinh x)\right](1)}{\sinh^2 x} \\ &= \frac{(0)\sinh x - (\cosh x)(1)}{\sinh^2 x} \\ &= -\frac{\cosh x}{\sinh^2 x} = -\left(\frac{\cosh x}{\sinh x}\right)\left(\frac{1}{\sinh x}\right) = -\coth x \operatorname{csch} x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) &= \frac{\left[\frac{d}{dx}(1)\right]\cosh x - \left[\frac{d}{dx}(\cosh x)\right](1)}{\cosh^2 x} \\ &= \frac{(0)\cosh x - (\sinh x)(1)}{\cosh^2 x} \\ &= -\frac{\sinh x}{\cosh^2 x} = -\left(\frac{\sinh x}{\cosh x}\right)\left(\frac{1}{\cosh x}\right) = -\tanh x \operatorname{sech} x\end{aligned}$$

$$\frac{d}{dx}(\coth x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\left[\frac{d}{dx}(\cosh x)\right]\sinh x - \left[\frac{d}{dx}(\sinh x)\right]\cosh x}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\operatorname{csch}^2 x$$